ECE 346: Digital Signal Processing — Spring 2021
Soo Min Kwon — Week of February 15, 2021

Recitation #4: Introduction to Sampling & Aliasing

Objective & Outline

The objective of this week's recitation session is introduce sampling, aliasing, and look at DSP systems (block diagrams) with C/D and D/C blocks. We will also plot these signals to get a better understanding of what is going on. The following is the outline of this solution guide:

1. Problems $1-4$: recitation problems
2. Problem 5: self-assessment problem

Problem 1 (Nyquist Sampling Theorem)**.** Recall that the Whittake-Shannon-Nyquist Sampling Theorem states that if we have a signal, say $x(t)$, that is bandlimited to $\Omega_{\max}$, then we should sample at a frequency Ω_s that satisfies

$$\Omega_s \geq 2 \cdot \Omega_{\max}. \tag{1}$$

With that said, determine the appriopriate Nyquist rate for sampling the following signals:

(a) $x_1(t) = 3\sin(120\pi t)$

(b) $x_2(t) = 3\sin(120\pi t) + 10\cos(250\pi t) - 2\cos(50\pi t)$

(c) $x_3(t) = \frac{\sin(50\pi t)}{\pi t}$

Now suppose there were two more CT signals, say $a(t)$ and $b(t)$ that were bandlimited to 120π and $50\pi\ \frac{\text{rad}}{\text{sec}}$, respectively. Determine the appriopriate Nyquist rate for sampling the following signals:

(a) $x_4(t) = a(t) * b(t)$

(b) $x_5(t) = a(t) \cdot b(t)$

Solution:

This problem is straightforward and so you should think of this as a warm-up problem:

(a) The signal $x_1(t)$ is bandlimited with $\Omega_{\max} = 120\pi$. Thus, we should sample at a sampling frequency of

$$\Omega_s \geq 2 \cdot \Omega_{\max} \tag{2}$$
$$\Omega_s \geq 240\pi. \tag{3}$$

Thus the Nyquist rate is $240\pi\ \frac{\text{rad}}{\text{sec}}$.

(b) The signal $x_2(t)$ has three sinuoids, where the maximum frequency is contained by $10\cos(250\pi t)$ with $\Omega_{\max} = 250\pi$. Thus with the same reasoning as the previous part, the Nyquist rate is $500\pi\ \frac{\text{rad}}{\text{sec}}$.

(c) We can look at the CTFT of $x_3(t)$:

$$X_3(j\Omega) = \begin{cases} 1, & |\Omega| \leq 50\pi \\ 0, & otherwise. \end{cases} \tag{4}$$

Of course, this is an ideal low pass filter with cutoff frequencies $[-50\pi, 50\pi]$. Thus, the Nyquist rate is $100\pi\ \frac{\text{rad}}{\text{sec}}$.

(d) We are given that that $a(t)$ and $b(t)$ are bandlimited to 60 Hz and 25 Hz, respectively. Using properties of convolution and multiplication:

$$x_4(t) = a(t) * b(t) \tag{5}$$
$$X_4(j\Omega) = A(j\Omega) \cdot B(j\Omega) \tag{6}$$

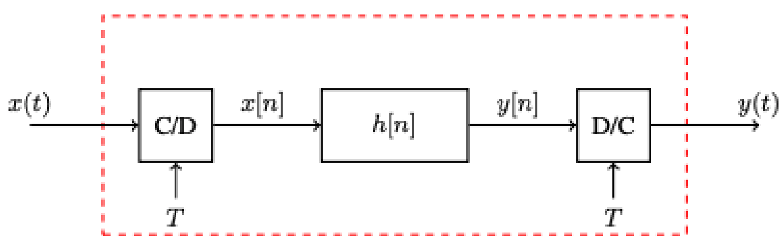
Thus, the bandwidth of this signal would be 25 Hz, and so the sampling frequency would be at least 50 Hz, or $100\pi\ \frac{\text{rad}}{\text{sec}}$.

(e) We can do the same reasoning as the previous part:

$$x_5(t) = a(t) \cdot b(t) \tag{7}$$
$$X_5(j\Omega) = \frac{1}{2\pi} A(j\Omega) * B(j\Omega). \tag{8}$$

This would yield a bandwidth of 85 Hz, and a sampling frequency of 170 Hz, or $340\pi\ \frac{\text{rad}}{\text{sec}}$.

Problem 2 (Sampling)**.** Consider the following block diagram,



where $T = 1/500$ seconds and the input to this diagram is

$$x(t) = 2\sin(100\pi t) - \cos(300\pi t) \tag{9}$$

and $h[n]$ is an ideal low pass filter with cutoff frequency $\omega_c = \frac{2\pi}{5}\ \frac{\text{rad}}{\text{sec}}$.

(a) State and plot the continuous-time Fourier transform of $x(t)$, $X(j\Omega)$.

(b) State and plot the discrete-time Fourier transform of $x[n]$, $X(e^{j\omega})$.

(c) State and plot the discrete-time Fourier transform of $y[n]$, $Y(e^{j\omega})$.

(d) State and plot the continuous-time Fourier transform of $y(t)$, $Y(j\Omega)$.

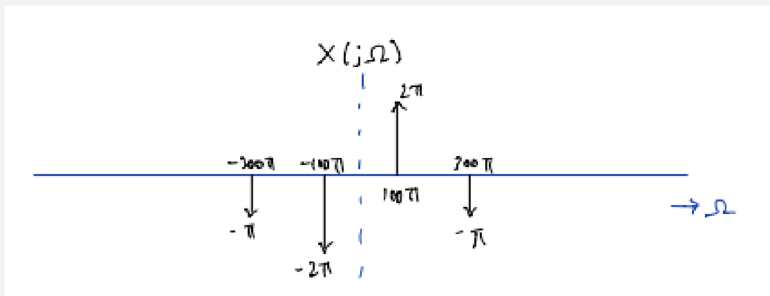
Solution:

(a) Hopefully by now, taking the CTFT of a signal like $x(t)$ should be easy. If not, I highly encourage you to come to office hours! In case you are not familiar, we can take the following steps:

$$x(t) = \frac{1}{j}e^{100\pi t} - \frac{1}{j}e^{-100\pi t} - \frac{1}{2}e^{j300\pi t} - \frac{1}{2}e^{j300\pi t} \tag{10}$$

$$X(j\Omega) = \frac{2\pi}{j}\delta(\Omega - 100\pi) - \frac{2\pi}{j}\delta(\Omega + 100\pi) - \pi\delta(\Omega - 300\pi) - \pi\delta(\Omega + 300\pi) \tag{11}$$

The plot of this would be



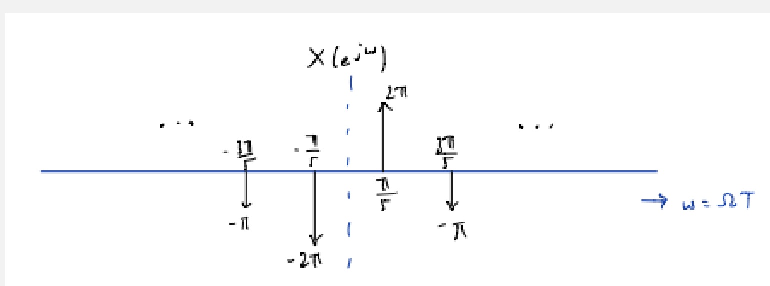
(b) For this part, there is actually a trick from going from the CTFT to the DTFT explicitly, and so plotting $X(j\Omega)$ first (previous part) and then plotting $X(e^{j\omega})$ would be easier. Then we can plot this simply by re-scaling the axis from part (a) with $\omega = \Omega T$, where $T = 1/500$ seconds. However, for now, we will just compute $x[n] = x(nT)$ and plot the DTFT. This will be

$$x[n] = 2\sin(\pi n/5) - \cos(3\pi n/5) \tag{12}$$

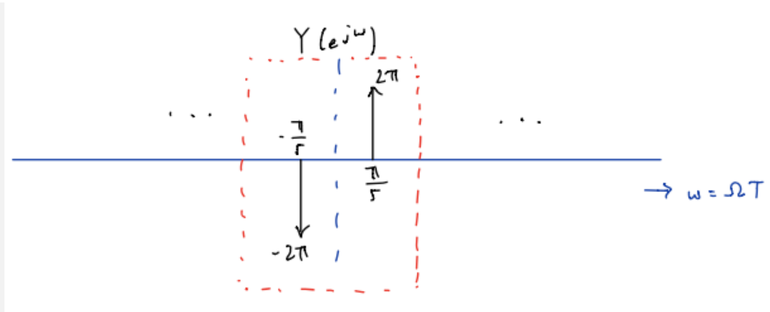
Taking the DTFT of this would yield

$$X(e^{j\omega}) = \frac{2\pi}{j}\delta(\omega - \pi/5) - \frac{2\pi}{j}\delta(\omega + \pi/5) - \pi\delta(\omega - 3\pi/5) - \pi\delta(\omega + 3\pi/5). \tag{13}$$

with the plot:



(c) To get $y[n]$, our input signal $x[n]$ goes through an ideal low pass filter with cutoff frequency $\omega_c = \frac{2\pi}{5}\ \frac{\text{rad}}{\text{sec}}$. I think plotting first and then stating the resulting equation would be easier:



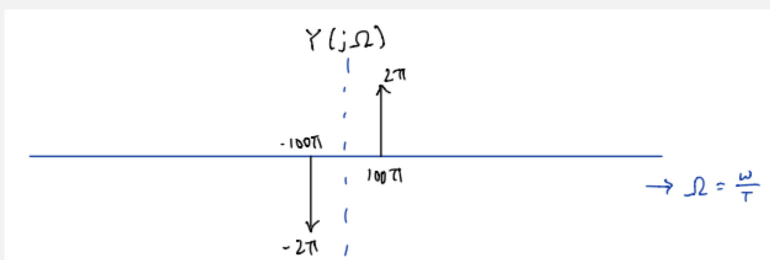
The low pass filter effectively filtered out the part of the signal with frequencies $\omega = \pm\frac{3\pi}{5}$. Our resulting signal is

$$Y(e^{j\omega}) = 2\pi\delta(\omega - \pi/5) - 2\pi\delta(\omega + \pi/5). \tag{14}$$

(d) There is also a trick to going from the DTFT to the CTFT plot, but for now we will do it explicitly:

$$y(t) = 2\sin(100\pi t) \tag{15}$$
$$Y(j\Omega) = 2\pi\delta(\omega - \pi/5) - 2\pi\delta(\omega + \pi/5), \tag{16}$$

with the final resulting plot:



Problem 3 (More Sampling)**.** Consider the continuous-time signal

$$x(t) = \cos(14\pi t) \tag{17}$$

that is sampled using a sampling frequency of $f_s = 10$ Hz. Let the signal $x_p(t)$ denote the impulse sampled signal:

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT). \tag{18}$$

(a) Provide a labeled sketch of the CTFT of $x_p(t)$.

(b) Suppose the signal $x_p(t)$ went through an ideal reconstruction filter (i.e. a LPF with cutoff frequency $\Omega = 10\pi$ with gain T). Sketch the resulting signal, $X_f(j\Omega)$.

(c) Provide a closed-form expression for $x_f(t)$. What happened to this signal?

Solution:

(a) Let's first derive $X_p(j\Omega)$:

$$x_p(t) = x(t) \cdot p(t) \tag{19}$$

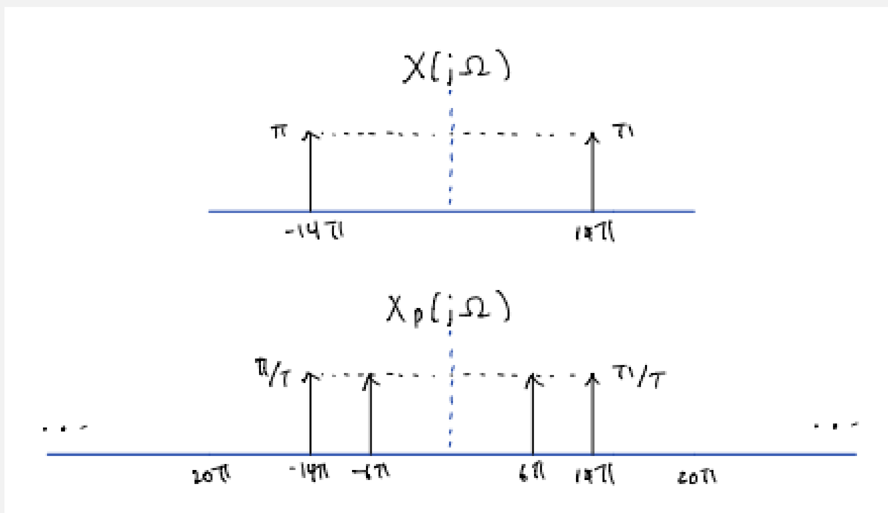
$$X_p(j\Omega) = \frac{1}{2\pi} X(j\Omega) * P(j\Omega) \tag{20}$$

$$X_p(j\Omega) = \frac{1}{2\pi} X(j\Omega) * \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\Omega - n\Omega_T) \tag{21}$$

$$X_p(j\Omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\Omega) * \delta(\Omega - n\Omega_T) \tag{22}$$

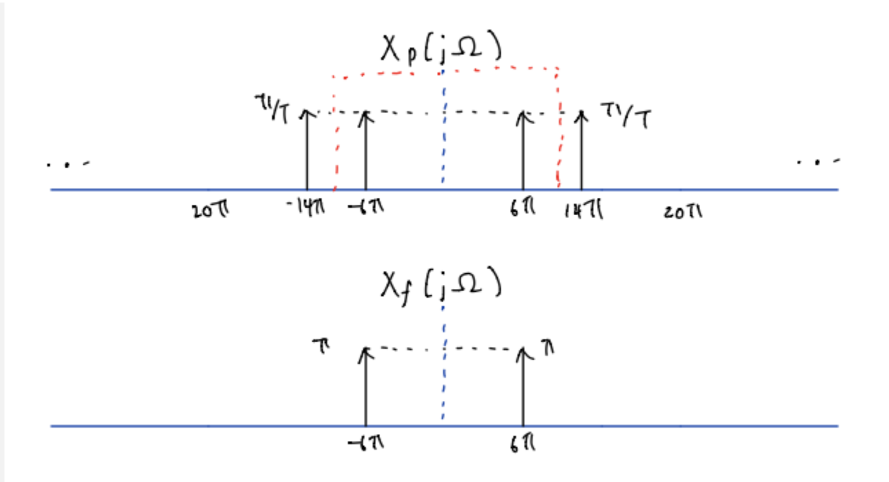
$$X_p(j\Omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j(\Omega - n\Omega_T)) \tag{23}$$

This is effectively taking $X(j\Omega)$ and repeating them every $\Omega_T = 2\pi f_s$:



(b) The signal $X_p(j\Omega)$ is going through an ideal reconstruction filter, $H_r(j\Omega)$ that is the following:

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \leq \pi/T \\ 0, & otherwise. \end{cases} \tag{24}$$



(c) From the plot in the previous part, we can conclude that $x_f(t) = \cos(6\pi t)$. This was an example of folding, where the frequencies were aliased and folded back onto $\Omega = 6\pi$.

Problem 4 (Even More Sampling)**.** Consider the continuous-time signal s(t) that is given by the following equation:

$$s(t) = 2\cos(250\pi t) + 5\sin(600\pi t + \pi/3). \tag{25}$$

Suppose the signal $s(t)$ was sampled with a sampling frequency of $f_s = 500$ Hz.

(a) Provide labeled plots of the magnitude and phase of the CTFT of $s(t)$, $S(j\Omega)$.

(b) Now consider the impulse-sampled signal

$$s_p(t) = \sum_{n=-\infty}^{\infty} s(t)\delta(t - nT), \tag{26}$$

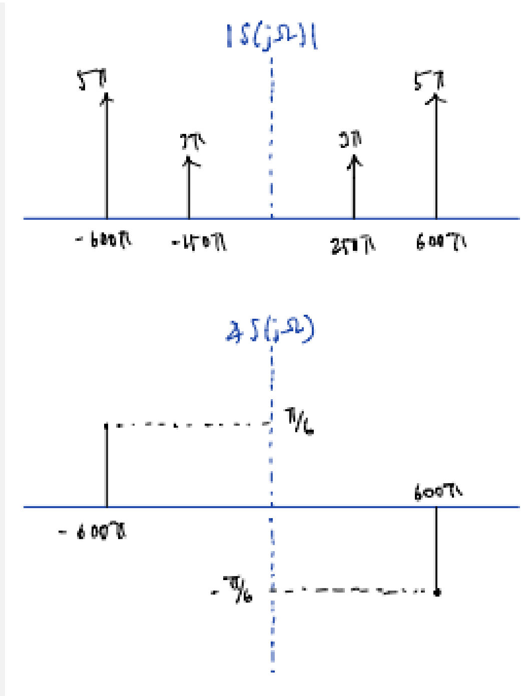
where $T = 1/f_s$. Provide labeled plots of the magnitude and phase of the CTFT of $s_p(t)$, $S_p(j\Omega)$.

Solution:

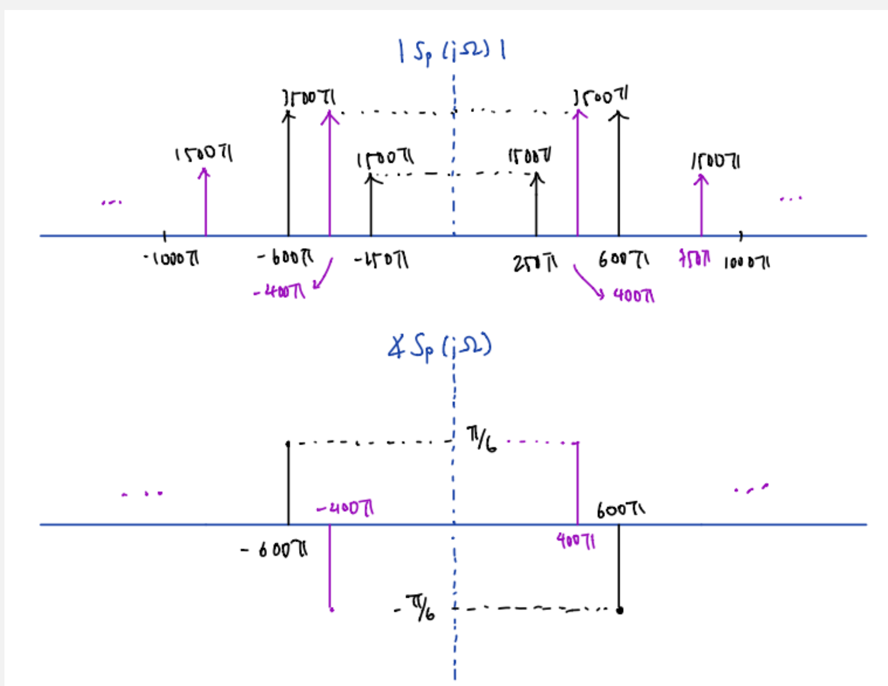
(a) Let's Euler-ize:

$$\begin{aligned}
s(t) &= 2\cos(250\pi t) + 5\sin(600\pi + \pi/3) & (27)\\
&= e^{j250\pi t} + e^{-j250\pi t} + \frac{5}{2j}\left[e^{j600\pi t}\cdot e^{j\pi/3} - e^{-j600\pi t}\cdot e^{-j\pi/3}\right] & (28)\\
&= e^{j250\pi t} + e^{-j250\pi t} + \frac{5}{2}\left[e^{j600\pi t}\cdot e^{-j\pi/6} - e^{-j600\pi t}\cdot e^{-j5\pi/6}\right] & (29)\\
&= e^{j250\pi t} + e^{-j250\pi t} + \frac{5}{2}\left[e^{j600\pi t}\cdot e^{-j\pi/6} + e^{-j600\pi t}\cdot e^{j\pi/6}\right] & (30)
\end{aligned}$$

$$S(j\Omega) = 2\pi[\delta(\Omega - 250\pi) + \delta(\Omega + 250\pi)] + 5\pi[\delta(\Omega - 600\pi)e^{-j\pi/6} + \delta(\Omega + 600\pi)e^{j\pi/6}] \tag{31}$$



(b) For this part, we can just look at the previous part for the plot of the magnitude, $|S(j\Omega)|$, and create samples every 1000π and scale the amplitude by $1/T$:



Remember that the height of the phases do not change from the reconstruction filter.

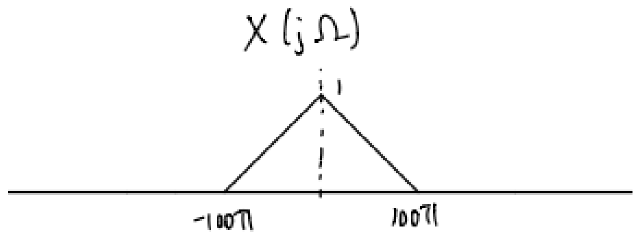
Problem 5 (Self-assessment)**.** Try to solve each problem by yourself first, and then discuss with your group.

1. **Reconstruction.** Recall the impulse-sampled signal $s_p(t)$ from Problem 4. Now suppose $s_p(t)$ was passed through an ideal reconstruction filter with the same sampling frequency ($f_s = 500$ Hz) and let $y(t)$ be the output of this filter.

   (a) Provide labeled plots for the magnitude $|Y(j\Omega)|$ and the phase $\angle Y(j\Omega)$.

   (b) Provide a closed-form expression of the reconstructed signal, $y(t)$.

2. **Sampling.** Suppose we had a signal $x(t)$ with CTFT as defined by the plot



   with a sampling frequency of $f_s = 75$ Hz.

(a) Let $x_p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)$ denote the impulse-sampled signal of $x(t)$. Provide a labeled sketch of the CTFT of $x_p(t)$, $X_p(j\Omega)$.

(b) Suppose the impulse-sampled signal is passed through an ideal reconstruction filter corresponding to the sampling frequency of $f_s = 75$ Hz. Plot the $Y(j\Omega)$, the output of this reconstruction filter.

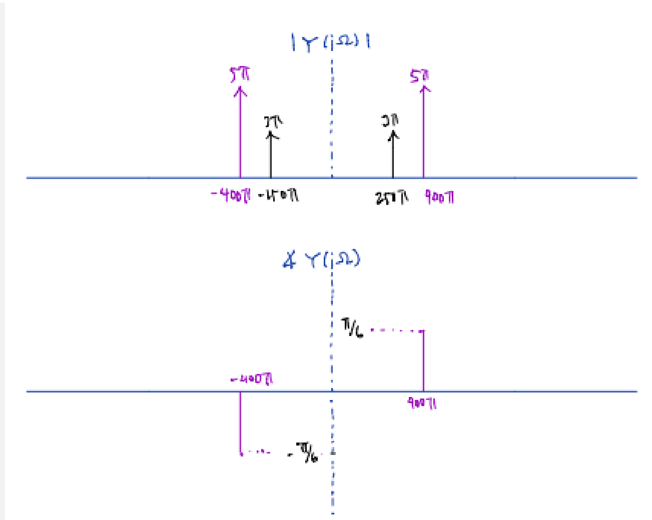
Solution:

1. Recall that the reconstruction filter would be

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \leq \pi/T \\ 0, & \textit{otherwise}, \end{cases} \tag{32}$$

   where $T = 1/500$.

   (a) So the magnitude and phase plots would be

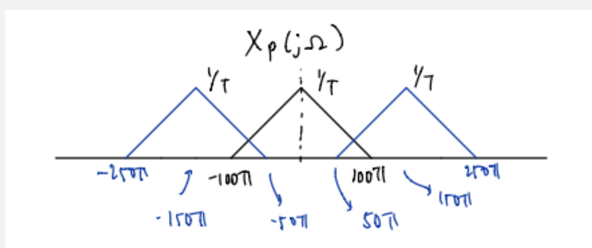


(b) We can obtain the closed-form expression by looking at the plot:

$$Y(j\Omega) = 2\pi[\delta(\Omega - 250\pi) + \delta(\Omega + 250\pi)] + 5\pi[\delta(\Omega - 400\pi)e^{j\pi/6} + \delta(\Omega + 400\pi)e^{-j\pi/6}] \quad (33)$$

$$y(t) = 2\cos(250\pi t) - 5\sin(400\pi t - \pi/3). \quad (34)$$

2. (a) Very similar question to what we've been looking at so far in this recitation session:



(b) The output will be different from the input as we have aliasing: